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A Note on Mean-Field Behavior for Self-Avoiding Walk on Branching Planes

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We consider the critical behavior of the susceptibility of the self-avoiding walk on the graph $\mathbf{T}\times\mathbf{Z}$, where $\mathbf{T}$ is a Bethe lattice with degree k and $\mathbf{Z}$ is the one dimensional lattice. By directly estimating the two-point function using a method of Grimmett and Newman, we show that the bubble condition is satisfied when $k>2$, and therefore the critical exponent associated with the susceptibility equals 1.

KEY WORDS: Self-avoiding walk; critical exponents.

1. INTRODUCTION AND STATEMENT OF RESULT

The mean-field critical behavior for the self-avoiding walk on the d-dimensional hypercubic lattice $\mathbf{Z}^d$ has been established by Hara and Slade[5] for $d\geqslant 5$. The method that they used is the lace expansion, which was introduced and used by Brydges and Spencer[2] to prove mean-field critical behavior for the weakly self-avoiding walk in more than four dimensions. In this note, we will consider the self-avoiding walk on *branching planes*. Models such as percolation, random cluster models as well as Ising spin systems on branching planes have been studied and found to exhibit interesting multiple phase transitions.[3,12] The lace expansion has proved to be a very successful method in establishing mean-field critical behavior for stochastic geometric models such as the self-avoiding walk, lattice trees and animals, and percolation on $\mathbf{Z}^d$ when d is above the corresponding upper critical dimensions.[2,5–8,13] But the lace expansion cannot be directly used on the branching planes, since some estimations in the expansion are based on the Fourier transform (and its inverse transform), and the usual Fourier

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transform cannot be directly used on the branching planes. In this note we will estimate the *two-point function* using a method of Grimmett and Newman[3] and then prove that the bubble condition is satisfied. To make our statement more precise, we first describe the lattice and the model. The lattice of branching planes in question is the direct product $\mathbf{T} \times \mathbf{Z}$, where $\mathbf{T}$ is a Bethe lattice with degree k (i.e., each site of $\mathbf{T}$ has exactly $k+1$ neighbors, where $k \geq 2$), and $\mathbf{Z} = \{\cdots, -1, 0, 1, \cdots\}$ is the one-dimensional lattice. The name branching planes comes from the fact that $\mathbf{T} \times \mathbf{Z}$ consists of branching planes, each of them a direct product of an infinite path of $\mathbf{T}$ and $\mathbf{Z}$. (Each horizontal layer of $\mathbf{T} \times \mathbf{Z}$ is a Bethe lattice.) A site of $\mathbf{T} \times \mathbf{Z}$ is denoted by (t, z), where the first component t is a site of $\mathbf{T}$ and the second component z is a site of $\mathbf{Z}$. The distance between two sites t_1 and t_2 of $\mathbf{T}$ is denoted by $|t_1 - t_2|$ and is defined to be the number of steps from t_1 to t_2 along the unique path between them. The distance between any two sites (t_1, z_1) and (t_2, z_2) of $\mathbf{T} \times \mathbf{Z}$ is then defined to be $|(t_1, z_1) - (t_2, z_2)| \equiv |t_1 - t_2| + |z_1 - z_2|$, where the second term in the sum is the absolute value of $z_1 - z_2$. Let σ be a distinguished site of $\mathbf{T}$ that we will call the origin of $\mathbf{T}$; then the origin of $\mathbf{T} \times \mathbf{Z}$ will be denoted by $(\sigma, 0)$. The distance between (t, z) and $(\sigma, 0)$ will be simply written as $|(t, z)|$.

An n-step self-avoiding walk ω on $\mathbf{T} \times \mathbf{Z}$ is an ordered set $\omega = (\omega(0), \omega(1),..., \omega(n))$ in $\mathbf{T} \times \mathbf{Z}$, with each $\omega(i) \in \mathbf{T} \times \mathbf{Z}$, $|\omega(i) - \omega(i+1)| = 1$, and $\omega(i) \neq \omega(j)$ for $i \neq j$. We write $|\omega| = n$ to denote the length of ω. Unless otherwise indicated, we take $\omega(0) = (\sigma, 0)$. We denote by c_n the number of n-step self-avoiding walks, and for $(t, z) \in \mathbf{T} \times \mathbf{Z}$ we denote by $c_n(t, z)$ the number of n-step self-avoiding walks for which $\omega(n) = (t, z)$. By convention, $c_0 = 1$ and $c_0(t, z) = \delta_{(t,z),(\sigma,0)}$. The existence of the *connective constant*

$$\mu = \lim_{n \to \infty} c_n^{1/n} \tag{1}$$

can be shown using the subadditivity argument[4,10,11]

Given a site (t, z) in $\mathbf{T} \times \mathbf{Z}$, the *two-point function* [between (t, z) and the origin $(\sigma, 0)$] is the generating function for the sequence $c_n(t, z)$, i.e.,

$$G_\lambda(t, z) = \sum_{n=0}^{\infty} c_n(t, z)\, \lambda^n = \sum_{\omega: (\sigma, 0) \to (t, z)} \lambda^{|\omega|} \tag{2}$$

The sum over ω is the sum over all self-avoiding walks, of arbitrary length $|\omega|$, which begin at the origin $(\sigma, 0)$ and end at (t, z). The *susceptibility* is then defined by

$$\chi(\lambda) = \sum_{(t,z) \in \mathbf{T} \times \mathbf{Z}} G_\lambda(t, z) \tag{3}$$

which can be written as the generating function of the sequence c_n, i.e.,

$$\chi(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n = \sum_{\omega} \lambda^{|\omega|} \tag{4}$$

where the sum over ω is the sum over all self-avoiding walks, of arbitrary length $|\omega|$, which begin at the origin. The power series in (4) has radius of convergence

$$\lambda_c \equiv (\lim_{n \to \infty} c_n^{1/n})^{-1} = 1/\mu \tag{5}$$

We will refer to λ_c as the *critical point*.

It is expected that

$$c_n \simeq \mu^n n^{\gamma - 1} \qquad \text{as} \quad n \to \infty \tag{6}$$

$$\chi(\lambda) \simeq (\lambda_c - \lambda)^{-\gamma} \qquad \text{as} \quad \lambda \uparrow \lambda_c \tag{7}$$

where γ is called the *critical exponent*, and the relation $f(x) \simeq g(x)$ as $x \to x_0$ means that there are positive constants c_1 and c_2 such that

$$c_1\, g(x) \leqslant f(x) \leqslant c_2\, g(x) \tag{8}$$

uniformly for x near its limiting value x_0.

We define the *bubble diagram* as

$$B(\lambda) = \sum_{(t,z) \in \mathbf{T} \times \mathbf{Z}} G_\lambda(t, z)^2 \tag{9}$$

The *bubble condition* states that the bubble diagram is finite at the critical point, i.e.,

$$B(\lambda_c) < \infty \tag{10}$$

In this note, we will prove that the power law (7) is valid with $\gamma = 1$ when the degree k of $\mathbf{T}$ is greater than 2. In particular we will prove that the bubble condition (10) is satisfied when $k > 2$. The bubble condition has been shown to imply (7) by Bovier *et al.*[1] (see also Section 1.5 of ref. 11). We believe that (7) holds for $k \geqslant 2$, although at the present our proof does not work for the $k = 2$ case. This is because we do not have a good upper bound for λ_c when $k = 2$. Currently we have no proof of power law behavior (6) for the microcanonical (as opposed to canonical) ensemble quantity c_n.

Theorem. For self-avoiding walk on $\mathbf{T} \times \mathbf{Z}$, if

$$\lambda\{1+\lambda+[2\lambda(1+\lambda)]^{1/2}\}/(1-\lambda) < 1/\sqrt{k} \tag{11}$$

then the bubble diagram is finite, i.e.,

$$B(\lambda) < \infty \tag{12}$$

The theorem will be proved in the next section. Its consequence is stated and proved as follows.

Corollary. For self-avoiding walk on $\mathbf{T} \times \mathbf{Z}$ with $k > 2$, we have that

$$\chi(\lambda) \simeq (\lambda_c - \lambda)^{-\gamma} \qquad \text{as} \quad \lambda \uparrow \lambda_c \tag{13}$$

with $\gamma = 1$, where the relation "$\simeq$" is in the same sense as in (8).

Proof of the Corollary. It is not hard to see that $\lambda_c \leqslant 1/(k+1)$. To see this, simply count the number of walks in which each step is either in the positive coordinate direction of $\mathbf{Z}$ if it is a "vertical" step [a vertical step is a step from (t, z_1) to (t, z_2) with $|z_1 - z_2| = 1$], or in the direction away from the origin σ of $\mathbf{T}$ if it is a "horizontal" step [a horizontal step is one from (t_1, z) to (t_2, z) with $|t_1 - t_2| = 1$]. Such walks are necessarily self-avoiding, so

$$c_n \geqslant (k+2)(k+1)^{n-1}$$

which implies that $\lambda_c = 1/\mu \leqslant 1/(k+1)$. Define $f(\lambda)$ to be the function on the left-hand side of (11); then $f(\lambda)$ is an increasing function and hence $f(\lambda_c) \leqslant f(1/(k+1))$. It is not difficult to check by direct calculation that $f(1/(k+1)) < 1/\sqrt{k}$ when $k \geqslant 4$. For $k = 3$, λ_c is bounded from above by $\lambda_c(\mathbf{Z}^3)$, the critical point of self-avoiding walk on $\mathbf{Z}^3$. This is because $\mathbf{Z}^3$ can be regarded as a subgraph embedded in $\mathbf{T} \times \mathbf{Z}$ with $k = 3$. On the other hand, $\lambda_c(\mathbf{Z}^3) < 0.21872$ from ref. 9. Substituting the value 0.21872 into $f(\cdot)$ reveals that $f(0.21872) < 1/\sqrt{3}$. So

$$f(\lambda_c) < 1/\sqrt{k} \qquad \text{when} \quad k \geqslant 3$$

Therefore by the theorem, we have that $B(\lambda_c) < \infty$, which implies the corollary by refs. 1 and 11. ∎

In the proof of the corollary, we have actually shown that the bubble diagram is finite not just at λ_c, but a little past λ_c. This kind of phenomenon does not occur for the self-avoiding walk on the hypercubic lattice $\mathbf{Z}^d$. The critical two-point function on $\mathbf{Z}^d$ decays like $|x|^{-(d-2)}$ for $d>4$ (so it is square summable for $d>4$); however, as we will show in the next section, the critical two-point function on $\mathbf{T}\times\mathbf{Z}$ (when $k>2$) decays exponentially (at least along the $\mathbf{T}$ component).

2. PROOF OF THE THEOREM

We need to show that $B(\lambda)$ is finite when (11) is satisfied. We write $B(\lambda)$ as follows:

$$\begin{aligned} B(\lambda) &= \sum_{(t,z)\in \mathbf{T}\times\mathbf{Z}} G_\lambda(t,z)^2 \\ &= \sum_{t\in\mathbf{T}} \sum_{z\in\mathbf{Z}} \left(\sum_{\omega:(\sigma,0)\to(t,z)} \lambda^{|\omega|}\right)^2 \\ &\leqslant \sum_{t\in\mathbf{T}} \left(\sum_{z\in\mathbf{Z}} \sum_{\omega:(\sigma,0)\to(t,z)} \lambda^{|\omega|}\right)^2 \end{aligned} \tag{14}$$

The main task of our proof is to show that

$$\sum_{z\in\mathbf{Z}} \sum_{\omega:(\sigma,0)\to(t,z)} \lambda^{|\omega|} < c(f(\lambda))^{|t|} \tag{15}$$

where c is a constant, $f(\lambda)$ is the function on the left-hand side of (11), and (recall that) $|t|$ is the distance from t to the origin σ of $\mathbf{T}$. The proof of (15) will begin in the next paragraph. From (14) and (15), we have that

$$\begin{aligned} B(\lambda) &\leqslant c^2 \sum_{t\in\mathbf{T}} f(\lambda)^{2|t|} \\ &= c^2 \left[1 + \sum_{n=1}^{\infty} (k+1)\, k^{n-1} f(\lambda)^{2n} \right] \\ &< \infty \end{aligned}$$

when $f(\lambda) < 1/\sqrt{k}$, i.e., when (11) is satisfied.

We now turn to the proof of (15). We will follow the idea in the proof of Proposition 1 of ref. 3, although the argument in ref. 3 is for percolation. A self-avoiding walk ω from $(\sigma, 0)$ to (t, z) can be thought of proceeding as follows. Starting at $(s_0, 0) = (\sigma, 0)$, it proceeds along vertical steps to some (s_0, y_0), then along (one) horizontal step to (s_1, y_0) (where s_1 is a site in $\mathbf{T}$ adjacent to s_0), then along vertical steps to some (s_1, y_1), and so on until it arrives at some (s_n, y_{n-1}) (where $s_n = t$), and finally along vertical steps to $(s_n, y_n) = (t, z)$. Denote $\vec{s} = (s_0, s_1,..., s_n)$. We will think of $\vec{s}$ as an

n-step simple walk in **T** from σ to t (a simple walk is a walk with the self-avoiding condition removed). Notice that

$$|\omega| = n + |y_0| + |y_1 - y_0| + \cdots + |y_n - y_{n-1}|$$

So

$$\text{LHS of (15)} \leqslant \sum_{n=|t|}^{\infty} \sum_{\vec{s}} \lambda^n \sum_{y_0=-\infty}^{\infty} \cdots \sum_{y_n=-\infty}^{\infty} \lambda^{|y_0|} \lambda^{|y_1-y_0|} \cdots \lambda^{|y_n-y_{n-1}|}$$

$$= \sum_{n=|t|}^{\infty} \sum_{\vec{s}} \lambda^n \Lambda^{n+1} \tag{16}$$

where $\Lambda = \sum_{y=-\infty}^{\infty} \lambda^{|y|} = (1+\lambda)/(1-\lambda)$, and the sum over $\vec{s}$ is the sum over all simple walks on **T**, of length n, which begin at σ and end at t.

We will improve the estimate in (16) as follows. For the aforesaid simple walk $\vec{s}$ of **T**, let $R(\vec{s})$ be the number of immediate reversals of $\vec{s}$. If the step from s_i to s_{i+1} is an immediate reversal (i.e., $s_{i-1} = s_{i+1}$), then $y_i \neq y_{i-1}$ since ω is self-avoiding, so that in (16), the Λ which corresponds to sum over y_i may be replaced by $\Lambda - 1$. It follows that

$$\text{LHS of (15)} \leqslant \sum_{n=|t|}^{\infty} \sum_{\vec{s}} \lambda^n (\Lambda - 1)^{R(\vec{s})} \Lambda^{n+1-R(\vec{s})}$$

$$= \Lambda \sum_{n=|t|}^{\infty} (\lambda\Lambda)^n \sum_{\vec{s}} (1 - \Lambda^{-1})^{R(\vec{s})} \tag{17}$$

Next we define each step of $\vec{s}$ to be either an *outstep* or *instep* (relative to the site t) according to the following rule. A step from s_i to s_{i+1} where $s_i \neq t$ is an outstep if and only if $|s_{i+1} - t| = |s_i - t| + 1$, and indicate an (arbitrary) step from t to one of its neighbors as an instep and the other k steps from t as outsteps. In this manner, there are exactly k possible outsteps and only one instep from each site of **T**. Among the n steps in $\vec{s}$, there are at most $(n - |t|)/2$ outsteps. This is because it takes $|t|$ insteps to reach t from σ, and among the remaining $n - |t|$ steps at most half are outsteps. We define $J_i = 1$ if the step from s_i to s_{i+1} is an instep and $J_i = 2$ otherwise. Set $T(\vec{J}) = \#\{i: J_{i-1} = 2,\ J_i = 1\}$ be the number of times an outstep is followed by an instep. Notice that $T(\vec{J}) \leqslant R(\vec{s})$. Each sequence $\vec{J} = (J_0, J_1, \ldots, J_n)$ of 1's and 2's corresponds to at most $k^{(n-|t|)/2}$ possible simple walks $\vec{s}$, since there are at most $(n - |t|)/2$ outsteps for each $\vec{s}$ and there are k possible ways to take each outstep. So from (17)

$$\text{LHS of (15)} \leqslant \Lambda \sum_{n=|t|}^{\infty} (\lambda\Lambda)^n \sum_{\vec{s}} (1 - \Lambda^{-1})^{T(\vec{J})}$$

$$\leqslant \Lambda \sum_{n=|t|}^{\infty} k^{(n-|t|)/2} (\lambda\Lambda)^n \sum_{\vec{J}} (1 - \Lambda^{-1})^{T(\vec{J})} \tag{18}$$

However,

$$\sum_{\bar{J}} (1-\Lambda^{-1})^{T(\bar{J})} = (1,1)\begin{pmatrix} 1 & 1 \\ 1-\Lambda^{-1} & 1 \end{pmatrix}^{n-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

which behaves for large n in the manner of $[1+(1-\Lambda^{-1})^{1/2}]^{n-1}$ where $1+(1-\Lambda^{-1})^{1/2}$ is the larger eigenvalue of the matrix on the right side of (19). Substituting (19) into (18), we have that

$$\begin{aligned} \text{LHS of (15)} &\leqslant c_1 \Lambda \sum_{n=|r|}^{\infty} k^{(n-|r|)/2}(\lambda\Lambda)^n[1+(1-\Lambda^{-1})^{1/2}]^{n-1} \\ &= c[f(\lambda)]^{|r|} \end{aligned}$$

where

$$f(\lambda) = \lambda\Lambda[1+(1-\Lambda^{-1})^{1/2}] = \lambda\{1+\lambda+[2\lambda(1+\lambda)]^{1/2}\}/(1-\lambda)$$

and

$$c = \frac{c_1\Lambda}{1+(1-\Lambda^{-1})^{1/2}} \sum_{n=0}^{\infty} [\sqrt{k} f(\lambda)]^n$$

is a finite constant when $f(\lambda) < 1/\sqrt{k}$. This completes the proof of (15) and hence the proof of the theorem. ▮

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